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Casualty Probabilities

of Gaussian Salvos

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Casualty Probabilities
of Gaussian Salvos

by

G. Trevor Williams

PREFACE

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This report is a generalization and extension of ORO-SP-24, "Optimum Dispersion for Gaussian Salvo." The first paper considered circular targets and circularly symmetrical aiming errors and salvo dispersions (X and Y components equal). This report introduces asymmetry in all three: elliptical targets and elliptical Gaussian distributions of aiming error and salvo dispersion.

The extension permits application to indirect as well as direct fire missions. The practical applications are currently under study and will be published later, together with a nomogram which permits ready application of the method.

Despite the general interest in this subject and the many closely related papers, the authors have found no other publication which provides the same results. After straightforward mathematical statement of the problem, this report succeeds by approximations in collapsing the eight major variables to a smaller number (mostly ratios) which are amenable to practical handling.

This report is separately issued to permit its use by others without delay, and to generate criticisms which might be useful in its use and interpretation.

SUMMARY

N independent shots, each of kill probability p and ballistic dispersions σ_x and σ_y, are fired with aiming errors τ_x and τ_y, at an elliptical target of semi-axes a_x and a_y. The probability K, that the target is killed is shown to be given approximately by

$$1 - K = \sqrt{\beta^2 - \gamma^2} \int_0^\infty I_0(\gamma u)\, e^{-\beta u - \alpha e^{-u}}\, du, \qquad (19)^*$$

where $I_0(\gamma u)$ is a Bessel function,

$$\alpha = \frac{Np\, a_x a_y}{2 S_x S_y}$$

$$\beta = \frac{1}{2}\left(\frac{S_x^2}{\tau_x^2} + \frac{S_y^2}{\tau_y^2}\right), \quad \gamma = \frac{1}{2}\left|\frac{S_x^2}{\tau_x^2} - \frac{S_y^2}{\tau_y^2}\right|, \qquad (50)$$

$$S_x^2 = \sigma_x^2 + \frac{1}{4} a_x^2, \quad S_y^2 = \sigma_y^2 + \frac{1}{4} a_y^2.$$

Equation (19) is simplified by a series of approximations to:

$$K = \Theta\, C(Y,Z) + (1 - \Theta)\, \phi\left(\sqrt{\beta \Theta \log \alpha}\right), \qquad (49)$$

where

$$\Theta = 1 - \frac{\gamma^2}{\beta^2} = \left[\frac{1}{2}\left(\frac{S_x \tau_y}{S_y \tau_x} + \frac{S_y \tau_x}{S_x \tau_y}\right)\right]^{-2} \qquad (48)$$

$$Y = \frac{1}{2\alpha} = \frac{S_x S_y}{Np\, a_x a_y},$$

$$Z = \frac{\beta}{2\alpha(\beta^2 - \gamma^2)} = \frac{S_x S_y}{2Np a_x a_y}\left(\frac{\tau_x^2}{S_x^2} + \frac{\tau_y^2}{S_y^2}\right) \qquad (32)$$

$C(Y, Z)$ is the incomplete gamma function and $\phi(x)$ is the central area of the normal distribution (interpreted as zero if $\alpha \ll 1$). In the instance that $\phi = 0$, it must also be assumed that $\Theta = 1.0$.

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*Equation numbers refer to the text.

CASUALTY PROBABILITIES OF GAUSSIAN SALVOS

Consider an elliptical target with semi-axes a_x and a_y. Let the aim-point be Gaussianly distributed about the target center with linear standard deviations τ_x and τ_y. A salvo of N shot is fired, each of which has, independent of the rest, a conditional probability p of incapacitating the target if it hits. Each of the shot is Gaussianly distributed about the actual aim-point with standard deviations σ_x and σ_y (so-called ballistic dispersions). The target axes and the principal axes of the two Gaussians are all assumed to be parallel. The problem is to express in a tractable form the probability, K, that the target will be incapacitated. A simpler version of this problem was treated in ORO-SP-24[1] with the restrictions $a_x = a_y = \rho$, $\sigma_x = \sigma_y = \sigma$, and $\tau_x = \tau_y = \tau$. The probability, $P(x,y)$, that a projectile aimed at (x,y) strikes the ellipse may be written

$$P(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \int_{\frac{\xi^2}{a_x^2} + \frac{\eta^2}{a_y^2} \leq 1} \int e^{-\frac{(\xi-x)^2}{2\sigma_x^2} - \frac{(\eta-y)^2}{2\sigma_y^2}} \, d\xi \, d\eta \tag{1}$$

To obtain an approximation to the above integral we replace the discrete target by a diffuse Gaussian target.[1] We write the Gaussian target as

$$\frac{C}{2\pi A_x A_y} e^{-\frac{x^2}{2A_x^2} - \frac{y^2}{2A_y^2}} \tag{2}$$

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and equate zero- and second-order moments with those of the ellipse. Since the Gaussian target distribution is centered about the origin the zero- and second-order moments are clearly

$$\mu_{00} = C, \mu_{20} = CA_x^2 \text{ and } \mu_{02} = CA_y^2 \tag{3}$$

For the ellipse, let $x = a_x r \cos \Theta$, $y = a_y r \sin \Theta$, $dxdy = a_x a_y r dr d\Theta$ and we find

$$\mu_{00} = \pi a_x a_y; \mu_{20} = \iint x^2 dxdy = a_x a_y \int_0^1 r\, dr \int_0^{2\pi} a_x^2 r^2 \cos^2 \Theta\, d\Theta$$

$$= \frac{\pi}{4} a_x^3 a_y \text{ and similarily } \mu_{02} = \frac{\pi}{4} a_x a_y^3 \tag{4}$$

and equating the two sets of moments (3) and (4) yields

$$C = \pi a_x a_y; A_x = \frac{1}{2} a_x; A_y = \frac{1}{2} a_y \tag{5}$$

and (2) becomes, on substituting from (5)

$$2 e^{-\frac{2x^2}{a_x^2} - \frac{2y^2}{a_y^2}} \tag{6}$$

Thus, for the sharp target, any fragment falling within the ellipse scores 1, a fragment falling outside scores 0, while for the diffuse Gaussian target a hit scores 2 at the origin and decreases to considerably less than 1 at the boundary of the ellipse as shown below:

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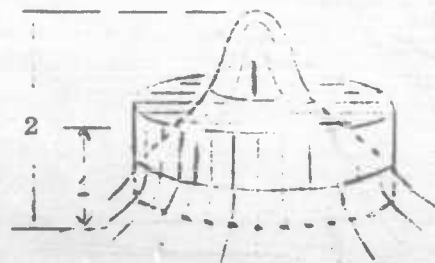
we now ask for the approximation to the probability given by (1), that a single shot aimed at the point (x,y) will hit the ellipse. Using the diffuse target (6), this may be written

$$P(x,y) = \frac{1}{\pi \sigma_x \sigma_y} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \, e^{-\left[\frac{(\xi-x)^2}{2\sigma_x^2} + \frac{(\eta-y)^2}{2\sigma_y^2} + \frac{2\xi^2}{a_x^2} + \frac{2\eta^2}{a_y^2}\right]} \,. \tag{7}$$

This immediately splits into the product of two integrals:

$$\frac{1}{\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} e^{-\left[\frac{(\xi-x)^2}{2\sigma_x^2} + \frac{2\xi^2}{a_x^2}\right]} d\xi \int_{-\infty}^{\infty} e^{-\left[\frac{(\eta-y)^2}{2\sigma_y^2} + \frac{2\eta^2}{a_y^2}\right]} d\eta \,;$$

the first integral can be written as

$$\int_{-\infty}^{\infty} e^{-\frac{\xi^2}{2\sigma_x^2} + \frac{x\xi}{\sigma_x^2} - \frac{x^2}{2\sigma_x^2} - \frac{2\xi^2}{a_x^2}} d\xi$$

$$= e^{-\frac{x^2}{2\sigma_x^2}} \int_{-\infty}^{\infty} e^{-\left(\frac{1}{2\sigma_x^2} + \frac{2}{a_x^2}\right)\xi^2 + \frac{x\xi}{\sigma_x^2}} d\xi$$

Now, it is well-known that

$$\int_{-\infty}^{\infty} e^{-a\xi^2 + b\xi} d\xi = \frac{1}{\sqrt{a}} e^{\frac{b^2}{4a}} \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$$

and the first integral becomes

$$= \sigma_x a_x \sqrt{\frac{\frac{\pi}{2}}{\sigma_x^2 + \frac{1}{4} a_x^2}} \, e^{-\frac{x^2}{2(\sigma_x^2 + \frac{1}{4} a_x^2)}} \,.$$

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This together with the corresponding expression involving y, reduces equation (7) to

$$P(x,y) = \frac{\frac{1}{2} a_x a_y}{\sqrt{(\sigma_x^2 + \frac{1}{4} a_x^2)(\sigma_y^2 + \frac{1}{4} a_y^2)}} \; e^{-\frac{x^2}{2(\sigma_x^2 + \frac{1}{4} a_x^2)} - \frac{y^2}{2(\sigma_y^2 + \frac{1}{4} a_y^2)}} \qquad (8)$$

Introducing the abbreviations $S_x^2 = \sigma_x^2 + \frac{1}{4} a_x^2$ and $S_y^2 = \sigma_y^2 + \frac{1}{4} a_y^2$ equation (8) becomes

$$P(x,y) = \frac{a_x a_y}{2 S_x S_y} \; e^{-\frac{x^2}{2 S_x^2} - \frac{y^2}{2 S_y^2}} \qquad (9)$$

Having assumed independence, the conditional probability, $K(x,y)$, that at least one of the N shot incapacitates the target is given by

$$K(x,y) = 1 - \left(1 - p\,P(x,y)\right)^N \qquad (10)$$

with p the conditional probability that a hit will be a casualty. If we make use of the Poisson approximation

$$1 - (1-x)^N \approx 1 - e^{-Nx} \quad \text{we may write (10) as}$$

$$K(x,y) \approx 1 - e^{-Np\,P(x,y)} \qquad (11)$$

The over-all casualty probability, K, is obtained by averaging (11) over the aim-point distribution :

$$K = \frac{1}{2\pi \, \tau_x \tau_y} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\tau_x^2}} \, dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\tau_y^2}} \, K(x,y) dy \qquad (12)$$

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Equations (9), (11) and (12) now yield

$$1 - K \approx \frac{1}{2\pi\tau_x\tau_y} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\tau_x^2}} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\tau_y^2}} dy \, e^{-\alpha e^{-\frac{x^2}{2S_x^2} - \frac{y^2}{2S_y^2}}} \tag{13}$$

where we have written, for brevity $\alpha = \dfrac{N p \, a_x a_y}{2 S_x S_y}$.

We now make the change of variable $x = S_x r \cos \Theta$, $y = S_y r \sin \Theta$,

$dxdy = S_x S_y \, r \, dr \, d\Theta$ and (13) becomes

$$1 - K \approx \frac{S_x S_y}{2\pi\tau_x\tau_y} \int_0^{\infty} e^{-\alpha e^{-\frac{r^2}{2}}} r \, dr \int_0^{2\pi} d\Theta e^{-\left(\frac{S_x^2}{\tau_x^2}\cos^2\Theta + \frac{S_y^2}{\tau_y^2}\sin^2\Theta\right)\frac{r^2}{2}} \tag{14}$$

One thing is immediately clear from (14); namely, that, although the problem
contains eight variables ($a_x, a_y, \sigma_x, \sigma_y, \tau_x, \tau_y$, N and p), the solution involves
only three, which might, for example, be taken to be α, $\dfrac{S_x}{\tau_x}$, $\dfrac{S_y}{\tau_y}$. We
shall soon find, however, that a slightly different choice for the latter two
variables will be advantageous. Putting $u = \frac{1}{2} r^2$ in (14) we find that

$$1 - K = \frac{S_x S_y}{2\pi\tau_x\tau_y} \int_0^{\infty} e^{-\alpha e^{-u}} du \int_0^{2\pi} d\Theta e^{-\left(\frac{S_x^2}{\tau_x^2}\cos^2\Theta + \frac{S_y^2}{\tau_y^2}\sin^2\Theta\right)u}$$

$$= \frac{S_x S_y}{2\pi\tau_x\tau_y} \int_0^{\infty} e^{-\alpha e^{-u} -\left(\frac{S_x^2}{\tau_x^2} + \frac{S_y^2}{\tau_y^2}\right)\frac{u}{2}} du \int_0^{2\pi} d\Theta e^{\left(\frac{S_y^2}{\tau_y^2} - \frac{S_x^2}{\tau_x^2}\right)\frac{u}{2}\cos 2\Theta} \tag{15}$$

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Now, for the Bessel function of imaginary argument, we have[2/]

$$I_0 (Z) = \frac{1}{2\pi} \int_0^{2\pi} e^{\pm z \cos \Theta} d\Theta .$$ (16)

Either sign in the exponent may be taken; in other words I_0 is an even function of Z. Therefore we may write

$$\beta = \frac{1}{2} \left(\frac{S_x^2}{\tau_x^2} + \frac{S_y^2}{\tau_y^2} \right) , \quad \gamma = \frac{1}{2} \left| \frac{S_x^2}{\tau_x^2} - \frac{S_y^2}{\tau_y^2} \right|$$ (17)

and we see at once that

$$\beta^2 - \gamma^2 = \frac{S_x^2 S_y^2}{\tau_x^2 \tau_y^2} .$$

Replacing Θ by $\frac{1}{2} \Theta$, (15) becomes

$$1 - K = \sqrt{\beta^2 - \gamma^2} \int_0^\infty e^{-\alpha e^{-u} - \beta u} du \frac{1}{4\pi} \int_0^{4\pi} e^{\pm \gamma u \cos \Theta} d\Theta .$$ (18)

By the periodicity of the cosine, we see on applying (16) to (18) that

$$1 - K = \sqrt{\beta^2 - \gamma^2} \int_0^\infty I_0 (\gamma u) e^{-\beta u - \alpha e^{-u}} du .$$ (19)

This is the desired probability, but it is a little complicated.

The discussion of (19) is facilitated by introducing an auxiliary quantity $\delta = \gamma / \beta$. From (17) it is clear that

$$0 \leq \gamma \leq \beta , \text{ hence } 0 \leq \delta \leq 1 .$$

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Indeed, it is well known[2] that

$$I_0 \ (z) \ \sim \ \frac{e^z}{\sqrt{2 \pi z}} \ (z \to \infty)$$

and consequently (19) does not converge unless $\gamma < \beta$. The case $\delta = 0$ has already been covered by a contour map.[1] One possibility for handling (19) might be to have it computed by the 1103 and similar contour maps drawn for, say 10 or 20 equally-spaced values of δ between 0 and 1. Such a port-folio of maps would not be particularly difficult to work with, although it would in general be necessary to read on the maps for two values of δ (or more, in those cases where K is a sensitive function of δ) and interpo-late to the exact value. The chief objection, however, is the prohibitive amount of labor required to draw up such a portfolio.

And, indeed, an over-scrupulous insistence on a precise evaluation of (19) is unwarranted, since (19) itself already rests on the diffuse target and Poisson approximations (cf.[1] where the ranges of validity of these are discussed). To approximate (19) by a simpler expression we shall first examine the behavior at $\delta = 0$ and fit for small δ with two degrees of freedom. We shall find this fit good except near $\delta = 1$. We shall then examine the behavior at $\delta = 1$ and again fit with two degrees of freedom. We then fit for all δ by a linear combination of the two approximations, thus adding another degree of freedom to our fit. The result is an expression which is simple to evaluate and which is highly accurate for all values of the parameters.

We shall require certain integrals:

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$$\int_0^\infty u^k \, du \, I_0 \, (\gamma u) \, e^{-u} = \int_0^\infty u^k e^{-u} \, du \, \frac{1}{2\pi} \int_0^{2\pi} e^{\gamma u \cos \Theta} \, d\Theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\Theta \int_0^\infty u^k e^{-u(1 - \gamma \cos \Theta)} \, du$$

$$= \frac{k!}{2\pi} \int_0^{2\pi} \frac{d\Theta}{(1 - \gamma \cos \Theta)^{k+1}} \tag{20}$$

Let $Z = e^{i\Theta}$, $\cos \Theta = \frac{1}{2} (Z + \frac{1}{Z})$ and then $dZ = i \, Z \, d\Theta$ and the integration over Z now runs along the unit circle in the complex plane and (20) becomes

$$\int_0^\infty u^k \, du \, I_0 \, (\gamma u) \, e^{-u} = \frac{k!}{2\pi i} \oint \frac{Z^k \, dZ}{(- \frac{\gamma}{2} Z^2 + Z - \frac{\gamma}{2})^{k+1}} \ . \tag{21}$$

The roots of the denominator of last integrand are given by

$$Z = \frac{1 \pm \sqrt{1 - \gamma^2}}{\gamma} \ .$$

Now, for the integrals to converge, we must have $\gamma \leq 1$, so that both expressions for Z are real. Since their product equals 1, one lies inside the unit circle (is less than 1) and one lies outside the unit circle (greater than 1). Only the former contributes its residue to the right-hand side of (21). We find then.

$$\int_0^\infty u^k \, du \, I_0 \, (\gamma u) \, e^{-u} = k! \left\{ \begin{array}{l} \text{residue at} \\ Z = \frac{1 - \sqrt{1 - \gamma^2}}{\gamma} \end{array} \right\} \text{ of } (\frac{Z^k}{(\frac{\gamma}{2} Z^2 + Z - \frac{\gamma}{2})^{k+1}}) \tag{22}$$

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Now let $\dfrac{1 - \sqrt{1 - \gamma^2}}{\gamma} = A$, $Z = W + A$,

$$Z - \frac{1 + \sqrt{1 - \gamma^2}}{\gamma} = W - \frac{2\sqrt{1 - \gamma^2}}{\gamma} = W - B$$

and the residue in (22) at $Z = A$ becomes

$$\mathop{\text{res}}_{(Z = A)} \frac{Z^k}{\left[-\dfrac{\gamma}{2}(Z-A)(Z-A-B) \right]^{k+1}}$$

$$= (-\frac{2}{\gamma})^{k+1} \mathop{\text{res}}_{W=0} \left[\frac{(W + A)^k}{W^{k+1}(W-B)^{k+1}} \right]$$

$$= (+\frac{2}{\gamma})^{k+1} \frac{A^k}{B^{k+1}} \mathop{\text{res}}_{W=0} \left[\frac{1}{W^{k+1}} (1 + \frac{W}{A})^k (1 - \frac{W}{B})^{-k-1} \right]$$

$$= (+\frac{2}{\gamma})^{k+1} \frac{A^k}{B^{k+1}} \mathop{\text{coeff}}_{W^k} \left[(1 + \frac{W}{A})^k (1 - \frac{W}{B})^{-k-1} \right]$$

$$= (+\frac{2}{\gamma})^{k+1} \frac{A^k}{B^{k+1}} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{1}{A^{k-\nu}} \binom{-k-1}{\nu} \frac{1}{(-B)^\nu}$$

$$= (+\frac{2}{\gamma B})^{k+1} \sum_{\nu=0}^{k} \frac{(k+\nu)!}{\nu!^2 (k-\nu)!} (\frac{A}{B})^\nu$$

and we obtain finally

$$\int_0^\infty u^k du \, I_0(\gamma u) \, e^{-u} = \frac{k!}{(1-\gamma^2)^{\frac{k+1}{2}}} \sum_{\nu=0}^{k} \frac{(k+\nu)!}{\nu!^2 (k-\nu)!} \left(\frac{1 - \sqrt{1-\gamma^2}}{2 \, \sqrt{1-\gamma^2}} \right)^\nu . \tag{23}$$

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We now wish to introduce the approximation

$$I_0 (\gamma u) \; e^{-\beta u} \approx D e^{-du} \tag{24}$$

To determine D and d we match zero and first-order moments. Now from (23) with $k = 0$, we find

$$\int_0^\infty du \; I_0 (\gamma u) \; e^{-\beta u} = \frac{1}{\beta} \frac{1}{\sqrt{1 - \frac{\gamma^2}{\beta^2}}} = \frac{1}{\sqrt{\beta^2 - \gamma^2}}$$

$$= \int_0^\infty D e^{-du} \; du = \frac{D}{d} \qquad \text{and therefore for the equality}$$

of the zeroth moments

$$\frac{1}{\sqrt{\beta^2 - \gamma^2}} = \frac{D}{d} \; . \tag{25}$$

By similar reasoning

$$\frac{\beta}{(\beta^2 - \gamma^2)^{3/2}} = \frac{D}{d^2} \tag{26}$$

for the equality of the first moments. Now solving (25) and (26) for d and D and substituting in (24) we obtain

$$I_0 (\gamma u) \; e^{-\beta u} \approx \frac{\sqrt{\beta^2 - \gamma^2}}{\beta} \; e^{-\frac{\beta^2 - \gamma^2}{\beta} u} \; . \tag{27}$$

Substituting (27) into (19) we obtain for small γ

$$1 - K \approx \frac{\beta^2 - \gamma^2}{\beta} \int_0^\infty du \; e^{-(\frac{\beta^2 - \gamma^2}{\beta}) u - \alpha e^{-u}} \; , \; (\gamma \approx 0) \tag{28}$$

Let us investigate how well (28) may be expected to approximate (19). When $\gamma = 0$ the two expressions are identical. Also, because of the two degrees of freedom in the fit, one anticipates that, as γ increases from 0 the

two solutions should diverge quite slowly from one another.

We now consider the behavior at $\gamma = \beta$; from (19) and (23) with $k = 0$ we find

$$1 - K = \sqrt{\beta^2 - \gamma^2} \int_0^\infty du\, I_0(\gamma u) \sum_{n=0}^\infty \frac{(-\alpha)^n}{n!} e^{-nu - \beta u}$$

$$= \sqrt{\beta^2 - \gamma^2} \sum_{n=0}^\infty \frac{(-\alpha)^n}{n!} \frac{1}{\sqrt{(\beta^2 + n)^2 - \gamma^2}} ,$$

so that

$$K \sim \sqrt{\beta^2 - \gamma^2} \sum_{n=1}^\infty (-1)^{n-1} \frac{\alpha^n}{n!} \frac{1}{\sqrt{n(n + 2\beta)}} , (\gamma \to \beta) . \qquad (29)$$

On the other hand, the approximation (28) yields

$$1 - K = \frac{\beta^2 - \gamma^2}{\beta} \int_0^\infty du\, e^{-\frac{\beta^2 - \gamma^2}{\beta} u} \sum_{n=0}^\infty \frac{(-\alpha)^n}{n!} e^{-nu}$$

$$= \frac{\beta^2 - \gamma^2}{\beta} \sum_{n=0}^\infty \frac{(-\alpha)^n}{n!} \frac{1}{n + \frac{\beta^2 - \gamma^2}{\beta}} ,$$

and so

$$K \sim \frac{\beta^2 - \gamma^2}{\beta} \sum_{n=1}^\infty \cdot (-1)^{n-1} \frac{\alpha^n}{n!\, n} , (\gamma \to \beta) .$$

Thus, both solutions approach zero as $\gamma \longrightarrow \beta$, but quite differently. It is also easily seen that the values of K given by (19) and (28) both approach zero as $\alpha \rightarrow 0$; indeed when α is small, the factor $e^{-\alpha e^{-u}}$ is nearly flat for all u and consequently the fit we have made in (27), which is perfect for a constant factor ($\alpha = 0$), should diverge quite slowly from (26) as α increases from 0. And when $\alpha \longrightarrow \infty$, both expressions for K approach 1.

When $\beta \longrightarrow 0$ the fact that $0 \leq \alpha \leq \beta$ shows that $\gamma \longrightarrow 0$ also, so the approximation is automatically good. For $\beta \longrightarrow \infty$, by replacing u by $\frac{u}{\beta}$ (19) may be written

$$1 - K = \sqrt{1 - \frac{\gamma^2}{\beta^2}} \int_0^\infty du \, I_0 \left(\frac{\gamma}{\beta} u\right) e^{-u-\alpha e^{-\frac{u}{\beta}}} .$$

I_0 is an even function of its argument; hence, retaining only first-order terms, we get

$$1 - K \sim \int_0^\infty du \, e^{-u-\alpha+\frac{\alpha u}{\beta}} = \frac{1}{1 - \frac{\alpha}{\beta}} e^{-\alpha}, (\beta \longrightarrow \infty) .$$

Likewise, by replacing u by $\frac{\beta}{\beta^2 - \gamma^2} u$ in (28), we get

$$1 - K = \int_0^\infty du \, e^{-u-\alpha \, e^{-\frac{\beta u}{\beta^2 - \gamma^2}}} ,$$

and retaining only first-order terms

$$1 - K \sim \int_0^\infty du \, e^{-u-\alpha+\frac{\alpha u}{\beta}} = \frac{1}{1 - \frac{\alpha}{\beta}} e^{-\alpha}, \quad (\beta \longrightarrow \infty) .$$

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Hence, (19) and (28) are identical to the first-order. Thus the fit is also very good for large β. Therefore, it is only for $\gamma \approx \beta$ that we need to expect substantial deviations of (28) from (19), and then only for intermediate values of α.

Reproduced in Figure 1 is the contour map given already in ORO SP-24[1] for the function we will denote here by

$$C(Y,Z) \equiv 1 - \int_0^\infty du\, e^{-u} - \frac{1}{2Y} e^{-\frac{Z}{Y}u} \quad . \tag{30}$$

Comparison of equations (28) and (30) show that we have for $\gamma \approx 0$

$$K \approx C \left(\frac{1}{2\alpha}, \frac{\beta}{2\alpha(\beta^2 - \gamma^2)} \right) \quad . \tag{31}$$

The values of Y and Z required to read the map of C (Y, Z) are, recalling that $\alpha = \dfrac{Np\, a_x\, a_y}{2\, S_X\, S_Y}$ and the values of β and γ from (17),

$$Y = \frac{S_X S_Y}{Np\, a_X\, a_y} \quad \text{and} \quad Z = \frac{S_X S_Y}{2\, Np\, a_X\, a_y} \left(\frac{\tau_X^2}{S_X^2} + \frac{\tau_y^2}{S_y^2} \right) \quad . \tag{32}$$

We pass on now to the fitting of (19) in the case where $\gamma \approx \beta$. Before we can do this we must study the behavior of (19) in this case. We have already seen that $K \longrightarrow 0$ when $\gamma \longrightarrow \beta$ for all α. However, we know that $K \longrightarrow 1$ as $\alpha \longrightarrow \infty$, no matter what are the values of β and γ. Hence, we anticipate a clash when α is large and $\gamma \approx \beta$. This situation is pointed up in (29), where the factor in front of the sum approaches 0 as $\gamma \longrightarrow \beta$, while the sum itself (it is not difficult to show) approaches ∞ as $\alpha \longrightarrow \infty$. One might be tempted to suggest that the sum in (29) be plotted up as a contour map in α and β and that (29) be used to approximate K when $\gamma \approx \beta$. This would mean, however, that by taking α sufficiently large, K would be
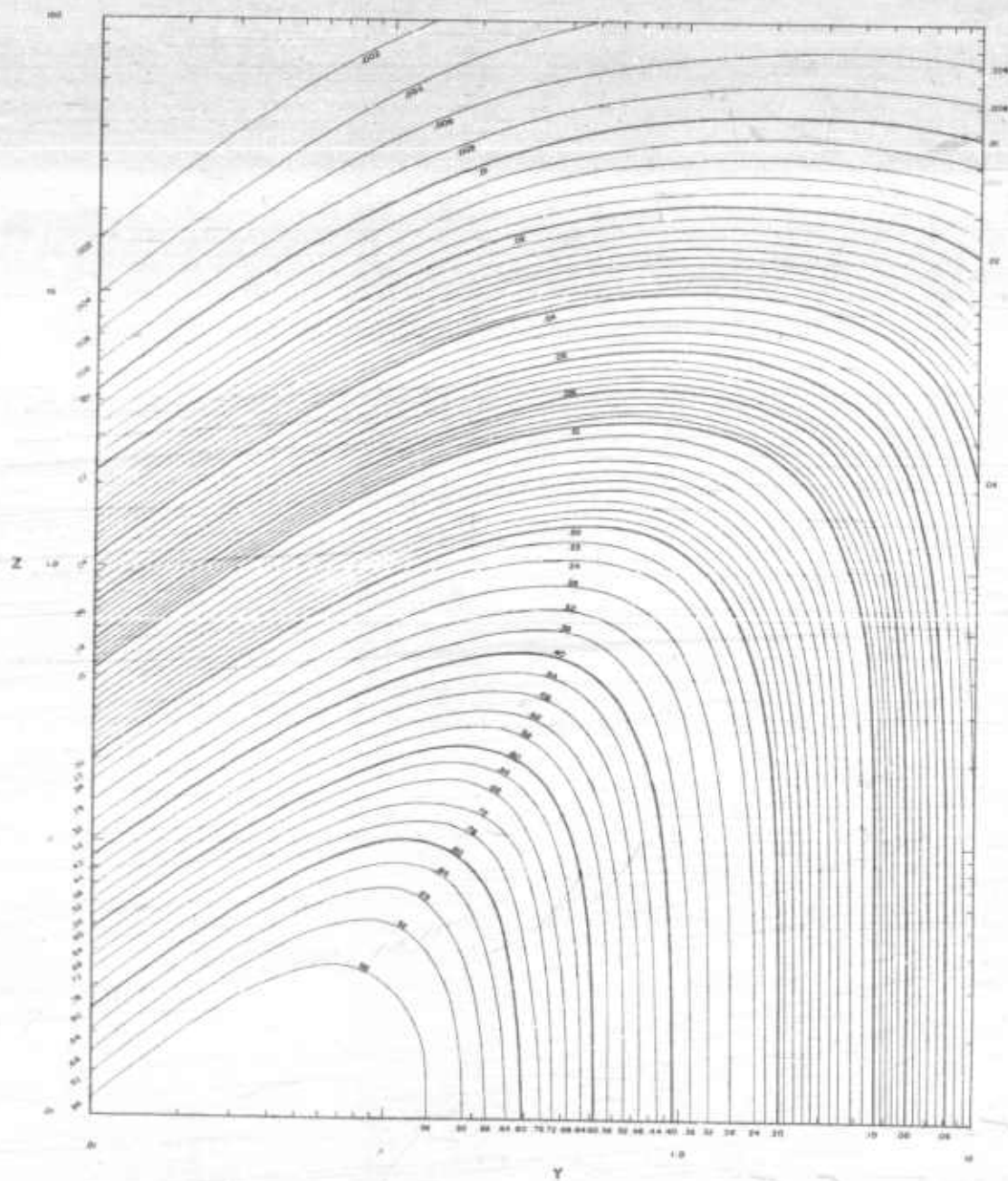
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Fig. 1—Graph of the C(Y, Z) Function

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estimated to be bigger than unity, and, indeed, as large as one pleases; this is not the sort of approximation we are looking for.

What is happening here may be understood by visualizing the sequence of contour maps as $\gamma \longrightarrow \beta$. The equiprobability contours keep moving out into the region of large α. Thus, for any given α, these contours continue to slip past as $\gamma \longrightarrow \beta$, so that, in the limit, the probability equals zero for all finite α. In order to stay in the region of non-zero probabilities, we must let α become large at the same time that we let $\gamma \longrightarrow \beta$. This is something we would do automatically if we were drawing up a portfolio of contour maps, since we would be guided not by the magnitude of α but by the desire to include all contours of reasonable probabilities.

Now the function $f(u) \equiv e^{-\beta u - \alpha e^{-u}}$, which occurs in (19) has a maximum, $e^{-\beta}(\frac{\beta}{\alpha})^{\beta}$, for $u = \log \frac{\alpha}{\beta}$. If we employ the fact that $I_0(Z) \sim \frac{e^Z}{\sqrt{2\pi Z}}$ $(Z \longrightarrow \infty)$ in (19) we find the integrand of (19) $\sim \frac{1}{\sqrt{2\pi \gamma u}} e^{-(\beta-\gamma)u - \alpha e^{-u}}$. If we ignore the slowly-varying factor, $\sqrt{u}$, here, which cannot shift the maximum by much, we find that the maximum of the integrand occurs at $u = \log \frac{\alpha}{\beta - \gamma}$. Hence when $\gamma \approx \beta$, this maximum will occur at a large value of u and thus only large u will contribute appreciably to the integral in (19). Therefore, the closer γ is to β the better the asymptotic expression $\frac{e^Z}{\sqrt{2\pi Z}}$ approximates the Bessel function so far as the integration in (19) is concerned. Making this approximation in (19) we have as $\gamma \longrightarrow \beta$:

$$1 - K \sim \sqrt{\frac{\beta^2 - \gamma^2}{2\pi \gamma}} \int_0^\infty \frac{1}{\sqrt{u}} e^{-(\beta-\gamma)u - \alpha e^{-u}} du \quad . \tag{33}$$

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(We see incidentally from this why (28) does not reproduce the behavior of (19) accurately when $\gamma \approx \beta$. For the behavior of $e^{-\beta u} I_0 (\gamma u)$ is dominated by the exponential factor $e^{-(\beta - \gamma) u}$ for large u, so long as γ differs appreciably from β; and hence (24) is a good approximation to use. When, however, $\gamma \approx \beta$, the exponential factor becomes of correspondingly less importance; the integration in (19) begins to be sensitive to the square root factor; and (24) is no longer a good approximation .)

Writing $u = Z^2$ in (33) we find

$$1 - K \sim \sqrt{\frac{\beta + \gamma}{2\gamma} \frac{\beta - \gamma}{\pi}} \int_0^\infty e^{-(\beta - \gamma) Z^2 - \alpha e^{-Z^2}} 2dZ, \text{ or}$$

$$1 - K \sim \sqrt{\frac{\beta - \gamma}{\pi}} \int_{-\infty}^\infty e^{-(\beta - \gamma) Z^2 - \alpha e^{-Z^2}} dZ, (\gamma \longrightarrow \beta), \tag{34}$$

since the integrand is now an even function of Z. In other words,

$$1 - K \sim < e^{-\alpha e^{-Z^2}} > \text{ where } Z = N (0, \frac{1}{2(\beta - \gamma)}), \tag{35}$$

where the corners indicate the expected value, and the latter half of the equation states that Z is normally distributed with zero mean and variance $\frac{1}{2(\beta - \gamma)}$. From this we see that when $\gamma \approx \beta$, the variance of Z is large and so the expection in (35) gives more and more weight to the wings of the function $e^{-\alpha e^{-Z^2}}$, i.e., since the latter approaches 1 as $Z \longrightarrow \infty$, this says that, for finite α, $1 - K \longrightarrow 1$ as $\gamma \longrightarrow \beta$, verifying what we noticed before.

It is convenient to rewrite (35) as

$$K \sim \ < 1 - e^{-\alpha e^{-z^2}} \ > \equiv \sqrt{\frac{\beta-\gamma}{\pi}} \int_{-\infty}^{\infty} \left(1-e^{-\alpha e^{-Z^2}}\right) e^{-(\beta-\gamma)Z^2} \, dZ \ . \quad (36)$$

If in the integral, we set $\gamma = \beta$, we find that it still converges; call it

$$j(\alpha) \equiv \int_{-\infty}^{\infty} \left(1-e^{-\alpha e^{-Z^2}}\right) dZ \quad . \quad\quad\quad\quad - (37)$$

At $Z = 0$, the integrand equals $1 - e^{-\alpha}$, and a 'typical' value of the integrand, somewhat analogous to the half life of a radioactive substance, will be gotten by setting $Z = \sqrt{\log \alpha}$. Thus, for large α, we may expect the area under the curve to be of the order of the magnitude of the height, $1 \cdot e^{-\alpha}$, times the typical width, $\sqrt{\log \alpha}$:

$$j(\alpha) \sim A \sqrt{\log \alpha} \quad (\alpha \longrightarrow \infty) \ .$$

Examination of the previous three equations shows that real and non-zero values of K are obtained only when

$$\alpha = \eta^{\frac{1}{\beta-\gamma}}, (\alpha > 1 \text{ and } \eta > 1) \ , \quad\quad\quad\quad (38)$$

where η is some new finite value. This is precisely the sort of behavior anticipated, for, when $\eta > 1$, $\alpha \to \infty$ as $\gamma \longrightarrow \beta$, which corresponds to the upward shifting of the probability contours described earlier.

Making the change of variable

$$Z = y\sqrt{\frac{\log \eta}{\beta - \gamma}}, \quad Z^2 = y^2 \ \frac{\log \eta}{\beta - \gamma}, \quad e^{-Z^2} = \eta^{-\frac{y^2}{\beta-\gamma}}$$

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in (36), and using (38), we obtain

$$K \sim \sqrt{\frac{\log \eta}{\pi}} \int_{-\infty}^{\infty} (1 - e^{-\eta^{\frac{1-y^2}{\beta-\gamma}}}) \; e^{-y^2 \log \eta} \; dy \quad .$$

Now, when $\eta > 1$, as we are assuming here,

$$1 - e^{-\eta^{\frac{1-y^2}{\beta-\gamma}}} \begin{array}{c} \longrightarrow \; 1 \;\; (y^2 < 1) \\ \longrightarrow \; 0 \;\; (y^2 > 1) \end{array} \left.\begin{array}{c} \\ \\ \end{array}\right\} \quad \text{as } \gamma \longrightarrow \beta \quad .$$

Thus

$$K \sim \sqrt{\frac{\log \eta}{\pi}} \int_{-1}^{1} e^{-y^2 \log \eta} \; dy, (\gamma \longrightarrow \beta) \quad . \tag{39}$$

Defining the cumulative normal distribution as usual by

$$\bar{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-\frac{1}{2}\xi^2} \; d\xi \tag{40}$$

(39) may be written

$$\begin{array}{c} K \sim \bar{\phi} \; (\sqrt{2 \log \eta}) \;\; (\eta > 1) \\ \sim 0 \qquad\qquad (\eta < 1) \end{array} \left.\begin{array}{c} \\ \\ \end{array}\right\} \quad \text{as } \gamma \longrightarrow \beta \quad . \tag{41}$$

Finally, recalling that $\eta = \alpha^{\beta-\gamma}$, our result becomes

$$\begin{array}{c} K \sim \bar{\phi}\left(\sqrt{2 \, (\beta-\gamma) \log \alpha}\right) \qquad (\alpha > 1) \\ \sim 0 \qquad\qquad\qquad\qquad\qquad (\alpha < 1) \end{array} \left.\begin{array}{c} \\ \\ \end{array}\right\} \quad \text{as } \gamma \longrightarrow \beta \quad . \tag{42}$$

It is now a very simple matter to improve our result by fitting (19) to it, analogously to the way we have already handled the case $\gamma \approx 0$. We want

$$e^{-\beta u} \; I_0 \; (\gamma u) \approx \frac{H}{\sqrt{u}} \; e^{-hu} \; , \tag{43}$$

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and equating zero and first-order moments, we find

$$\frac{1}{\sqrt{\beta^2-\gamma^2}} = H \int_0^\infty e^{-hu} \frac{du}{\sqrt{u}} = 2H \int_0^\infty e^{-hZ^2} dZ = H\sqrt{\frac{\pi}{h}} \; ,$$

$$\frac{\beta}{(\beta^2-\gamma^2)^{\frac{3}{2}}} = H \int_0^\infty \sqrt{u}\; e^{-hu}\, du = 2H \int_0^\infty Z^2 \; e^{-hZ^2} dZ = \frac{H}{2h}\sqrt{\frac{\pi}{h}} \; .$$

Solving for H and h, we get

$$h = \frac{\beta^2-\gamma^2}{2\beta} , \quad H = \frac{1}{\sqrt{2\pi\beta}} \quad .$$

Substituting in (43) gives

$$e^{-\beta u}\, I_0\,(\gamma u) \approx \frac{1}{\sqrt{2\pi\beta u}} \; e^{-\frac{\beta^2-\gamma^2}{2\beta} u}, \tag{44}$$

and substituting (44) in (19), we find

$$1 - K \approx \sqrt{\beta^2-\gamma^2} \int_0^\infty \frac{1}{\sqrt{2\pi\beta u}} \; e^{-\frac{\beta^2-\gamma^2}{2\beta} u}\, e^{-\alpha e^{-u}} du \qquad , \tag{45}$$
$$(\gamma \approx \beta)$$

and again writing $u = Z^2$

$$1 - K \approx \sqrt{\frac{\beta^2-\gamma^2}{2\pi\beta}} \int_{-\infty}^\infty e^{-\frac{\beta^2-\gamma^2}{2\beta} Z^2 - \alpha e^{-Z^2}} dZ \quad .$$

But this is precisely of the form of equation (34) with $\beta-\gamma$ replaced by $\frac{\beta^2-\gamma^2}{2\beta}$.

Hence we can write the solution from (42)

$$K \approx \Phi \left(\begin{array}{ll} \sqrt{\beta(1-\frac{\gamma^2}{\beta^2}) \log \alpha} & (\alpha > 1) \\ \approx 0 & (\alpha < 1) \end{array} \right\} \quad (\gamma \approx \beta) \quad . \tag{46}$$

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Finally, it remains to splice togther the two solutions (28) and (46). To do so, we now make the approximation

$$e^{-\beta u} I_0 (\gamma u) \approx \Theta \frac{\sqrt{\beta^2-\gamma^2}}{\beta} e^{-\frac{\beta^2-\gamma^2}{\beta} u} + (1-\Theta)\frac{1}{\sqrt{2\pi\beta u}} e^{-\frac{\beta^2-\gamma^2}{2\beta} u} \quad (47)$$

i.e., we make a linear combination of equations (27) and (44). Equality of the zero and first-order moments of both sides is guaranteed by the form of the expression on the right. Thus, to determine Θ, we match the second moments of both sides. We now use (23) with $k = 2$ and find

$$\frac{2\beta^2+\gamma^2}{(\beta^2-\gamma^2)^{\frac{5}{2}}} = \Theta \frac{2\beta^2}{(\beta^2-\gamma^2)^{\frac{5}{2}}} + (1-\Theta)\frac{3\beta^2}{(\beta^2-\gamma^2)^{\frac{5}{2}}} ,$$

from which it follows immediately that

$$\Theta = 1 - \frac{\gamma^2}{\beta^2} . \quad (48)$$

Substituting (47) and (48) into (19) and writing the answer in terms of (31) and (46) we obtain our final result

$$K \approx (1 - \frac{\gamma^2}{\beta^2}) C (\frac{1}{2\alpha},\frac{\beta}{2\alpha(\beta^2-\gamma^2)}) + \frac{\gamma^2}{\beta^2} \Phi (\sqrt{\beta (1 - \frac{\gamma^2}{\beta^2}) \log \alpha}) \quad (49)$$

where Φ is to be interpreted as zero if $\alpha < 1$. We now recall the definitions of quantities occuring in the equation in terms of the parameters of the original formulation as follows:

$$\alpha = \frac{Np \, a_x \, a_y}{2S_x \, S_y} , \quad \beta = \frac{1}{2}\left(\frac{S_x^2}{\tau_x^2} + \frac{S_y^2}{\tau_y^2}\right) \quad \text{and}$$

$$\gamma = \frac{1}{2}\left|\frac{S_x^2}{\tau_x^2} - \frac{S_y^2}{\tau_y^2}\right| . \quad (50)$$

Equation (32), which tells us where to read on Figure 1 is still valid; and we note also that the quantity $\Theta = 1 - (\frac{\gamma}{\beta})^2$ which occurs in two places in (49) is merely $\Theta = 1 - \frac{\gamma^2}{\beta^2} = \left[\frac{1}{2} \left(\frac{S_x \tau_y}{S_y \tau_x} + \frac{S_y \tau_x}{S_x \tau_y} \right) \right]^{-2}$. It should be recalled that ϕ is to be interpreted as zero if $\alpha \leq 1$. In this instance Θ must also be interpreted as equal to 1.0. Θ is defined as a weighting factor between two solutions, if one solution does not apply, the full value of the other solution must be used.

From the form of the weighting coefficients in (49) we see that for $\frac{\gamma}{\beta}$ small, the second term contributes very little to the answer; indeed the coefficients have a horizontal tangent at $\gamma = 0$. In the latter case, however, when $\gamma \approx \beta$, a slight drop in $\frac{\gamma}{\beta}$ produces a proportional admixture of the first term to the second term. In other words (31) remains accurate over a wider region than (46).

From the form of (38) we see that when $\gamma \approx \beta$ and α approaches 1, then η approaches 1, but still more rapidly. Since the validity of the function ϕ depends on η being more than 1, it is to be expected that the solution has a danger point that should not be closely approached. This turns out to be the case, and the critical region appears when ϕ exists and α is approximately 1. This situation is easily handled in practice by raising α somewhat, perhaps by increasing N, and then extrapolating back to the desired value of α.

An illustrative example of the application of the formulas developed in this paper is included in these final paragraphs. The particular formulas used in any application are scattered throughout the paper. The relations between S^2, σ^2, and a^2 were introduced immediately following equation (8). α, as a function of N, p, a_x, a_y, S_x, and S_y, was first introduced following equation (13) and restated in equation (50). β and γ are shown in equations (17) and (50). The $C(Y, Z)$ function is graphed in Figure 1; both Y and Z are defined in equation (32). One now requires only Θ and Φ to determine the probability of a casualty, K. Θ is defined in equation (48), and Φ is the central area of the Normal (Gaussian) distribution. The final probability of a casualty, K, is determined in equation (49).

A simple example is selected which affords a comparison with earlier work: A weapon that has a large dispersion characterized by range and deflection probable errors of 94 and 18 yards respectively is used to attack a small target 4 yards in radius (symmetrical target, $a_x = a_y$). A single hit destroys the target (p = 1). What is the probability of destruction for 32 shots for aiming error CEP's of 0, 50, 100, and 200 yards (symmetrical aiming error, $\tau_x = \tau_y$)? In converting from range and deflection probable errors to standard deviations, one obtains $\sigma_x = 140$ yards and $\sigma_y = 27$ yards. As a result

$$S_x^2 = 19,600$$

and

$$S_y^2 = 733.$$

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By the relations given in equation (17) and (50)

$$\alpha = 0.0677$$

and

$$Y = \frac{1}{2\alpha} = 7.38.$$

The relations involving β and γ both depend on the evaluation of τ, given in this example in terms of CEP's. Because of the symmetry, $\tau = \tau_x = \tau_y$. Once τ is determined, β, γ, Z, Θ, and $\sqrt{\beta \Theta \log_e \alpha}$ can be obtained. The following table shows these figures:

Table 1

PARAMETERS VS CEP

CEP	τ	β	γ	Z	Θ	$\sqrt{\beta \Theta \log_e \alpha}$
0	0	–	–	–	–	–
50	42	5.64	5.23	4.74	0.139	-2.05
100	85	1.41	1.31	18.8	0.139	-0.511
200	170	0.352	0.327	75.8	0.139	-0.128
300	255	0.157	0.145	171	0.139	-0.057

In each case $\sqrt{\beta \Theta \log_e \alpha}$ is imaginary, and we must take $\Phi = 0$. When $\sqrt{\beta \Theta \log_e \alpha}$ is positive, Φ is just the central area of Normal (Gaussian) distribution. When Φ is zero, Θ must be assumed equal to 1.00. In this example Φ is always zero, Θ is always 1.00, and K, the probability of a casualty, is just C(Y, Z). The results are given in the following table:

Table 2

PROBABILITY OF A CASUALTY FOR GIVEN CEP's

CEP (yds)	Probability of Casualty (%)	Earlier Results (%)
0	6.4	6.3
50	4.0	-
100	1.8	1.7
200	0.6	-

For comparison, values calculated in detail in an earlier work* are shown in the right-hand column.

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*ORO-T-304, "Comparisons of Cost among Tactical Air, Field Artillery, and Heavy Mortars (U)," Sep 55. SECRET-FORMERLY RESTRICTED DATA

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